Registration of electrode coordinates with fiducials
Thomas Ferree and Phan Luu
Revised 8/24/00

This document addresses the registration of electrode coordinates with head fiducial points. The objective is to put all such electrode files in a common framework to compare how the electrode montages sit upon the human head on average. What follows is a step-by-step sequence for doing this. The choices made are not necessarily unique, but are sensible given the stated objective. Note that the resulting coordinate system is not simply related to any best-fit spheres approximation of the head, as would be used in bioelectric field modeling for example.

1. A Polhemus Tracker or Photogrammetry system returns the position of each electrode and the fiducial points in some arbitrary coordinate system

\[ \vec{r}_i(p) = \left( x_i(p), y_i(p), z_i(p) \right)^T \]

where \( i=1,\ldots,129, L, R, N, V \). The first 129 points are the electrode locations. L and R represent the left and right pre-auricular points, N represents the nasion, and V represents the vertex. The vectors to each point exist in 3-dimensional space in a coordinate independent sense, but the superscript \( (p) \) indicates that here these are represented in the Polhemus coordinate system. The superscript T stands for transpose, and reflects our preference toward working with column vectors to facilitate the matrix multiplications below.

2. We conceive a fiducial coordinate system, denoted with a superscript \( (f) \), with the following properties. First, its x-axis lies along the line connecting L and R with the positive direction pointing toward R. Second, its y-axis is perpendicular to the line LR, and passes through the nasion in the positive direction. The x-axis and y-axis intersect at a point, which forms the center of this coordinate system, but with the constraints that: i) the y-axis pass through the nasion, and ii) the x-axis and y-axis meet at a right angle, the intersection will not in general be at the midline of the line LR. As a matter of convention, we are placing more emphasis on the nasion than on the midpoint of the line LR.

3. It is a simple problem in 3-dimensional Euclidean geometry to determine the equation for the unique line passing through the nasion N and intersecting the line LR at a right angle. Let us denote the intersection of these lines with the vector

\[ \vec{r}_C(p) = \left( x_C(p), y_C(p), z_C(p) \right)^T \]

where the subscript C stands for the center of the fiducial coordinate system in Polhemus coordinates \( (p) \). An expression for the location of the point C in a slightly simplified case is given by Strang (1980). In terms of the fiducial position vectors in the Polhemus coordinates, we have
4. The next step is to translate all our coordinates to be centered on the above point. This is done trivially with the transformation
\[
\hat{r}_C^{(p)} = \hat{r}_L^{(p)} + \frac{\left(\hat{r}_R^{(p)} - \hat{r}_L^{(p)}\right) \cdot \left(\hat{r}_N^{(p)} - \hat{r}_L^{(p)}\right)}{\left|\hat{r}_R^{(p)} - \hat{r}_L^{(p)}\right|^2} \left(\hat{r}_R^{(p)} - \hat{r}_L^{(p)}\right)
\]
where the superscript (c) denotes that the new vectors are in our so-called “centered” coordinate system. We of course must have the trivial result that
\[
\hat{r}_C^{(c)} = (0, 0, 0)^T
\]
that is, the center point C is at the origin of our centered coordinate system (c). We are still not done, however, because this centered coordinate system (c) is rotated arbitrarily relative to our desired fiducial coordinate system (f).

5. We can easily define unit vectors in these centered coordinates which lie along the axes of the desired fiducial coordinate system. First, the x-axis is defined to lie along the line connecting the center point C and the right pre-auricular point R:
\[
\hat{\chi}^{(c)} = \frac{\hat{r}_R^{(c)}}{|\hat{r}_R^{(c)}|}
\]
(Because the center point C is the origin of our centered coordinate system it does not appear explicitly in the above equation.) Second, the y-axis is defined to lie along the line connecting the center point C and the nasion N:
\[
\hat{\gamma}^{(c)} = \frac{\hat{r}_N^{(c)}}{|\hat{r}_N^{(c)}|}
\]
A useful step in verifying the computer implementation is to test that these two vectors are indeed orthogonal. Since they are unit vectors, this may be considered satisfied of their dot product is small compared to unit. Finally, the z-axis is defined to be perpendicular to the resulting xy-plane, with its positive direction determined as usual by the right hand rule:
\[
\hat{\zeta}^{(c)} = \hat{\chi}^{(c)} \times \hat{\gamma}^{(c)}
\]
By definition, the fiducial unit vectors have trivial representations in the fiducial coordinate system (f).
6. Our next goal is to determine a matrix which, when applied to the electrode positions in the centered coordinate system (c), will rotate them into the fiducial coordinate system (f). This is most easily done as follows. We first construct a matrix $M$, whose columns are the fiducial unit vectors in centered coordinates:

$$M = \begin{bmatrix} \hat{x}^{(c)}, \hat{y}^{(c)}, \hat{z}^{(c)} \end{bmatrix}$$

This matrix has the obvious property that, when applied to the fiducial unit vectors in the fiducial coordinate system (f), it translates them into the centered coordinate system (c):

$$\hat{x}^{(c)} = M \hat{x}^{(f)}$$
$$\hat{y}^{(c)} = M \hat{y}^{(f)}$$
$$\hat{z}^{(c)} = M \hat{z}^{(f)}$$

Since any vector in the fiducial coordinate system can be represented as a linear sum of the fiducial unit vectors, the transformation matrix $M$ actually rotates any vector from the fiducial coordinate system (f) to the centered coordinate system (c):

$$\vec{r}_i^{(c)} = M \vec{r}_i^{(f)}$$

Once this matrix is constructed it can be tested by applying it to each of the hat vectors in the (f) basis. It should be that the trivial hat vectors in the fiducial system get transformed to the nontrivial hat vectors in the centered coordinate system.

7. By definition, the inverse of $M$ rotates any vector in the centered coordinates (c) into the fiducial coordinates (f), that is

$$\vec{r}_i^{(f)} = M^{-1} \vec{r}_i^{(c)}$$

This is our desired result, and can be applied to each of the electrode and fiducial positions. Conveniently, it can be shown that the matrix $M$ is an orthogonal matrix whose inverse is equal to its transpose (Arfken and Weber, 1995).

$$M^{-1} = M^T$$
so the inverse of M can be gotten trivially. The final result may be tested by applying the inverse (transpose) matrix to each of the hat vectors in the centered coordinates (c) and verifying that they are properly rotated into the fiducial coordinate axes.

As a final test of the entire set of transformations, one can verify that the distance between each pair of electrodes is preserved. Given the deterministic nature of each of the transformation, this can be expected to be satisfied to within the accuracy of single or double precision. If the distance between each pair of electrodes is preserved, and right pre-auricular point and nasion lie on the fiducial x and y axes respectively, then the coordinate transformation must be correct.

References
