Weighted regularization in electrical impedance tomography with applications to cerebral stroke

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Abstract

We apply electrical impedance tomography to detect and approximately localize brain impedance changes. We assume that baseline conductivity parameters are known for the major head tissues, and focus on changes in the brain compartment only. Forward solutions are computed using the finite element method in two dimensions. We show that various four-electrode impedance measurement patterns, which are theoretically equivalent by the reciprocity theorem, have different sensitivities to the brain compartment, and this becomes a factor in the presence of measurement noise. We use singular value decomposition to motivate a weighting scheme which normalizes the average surface sensitivity to conductivity as a function of depth, and show that this scheme forces all measurement patterns to have similar overall sensitivity. Our main result is the observation that this weighting scheme improves image quality overall. We apply the method to detect and approximately localize conductivity changes, like those associated with ischemic and hemorrhagic stroke. Accurate localization will require more research to determine, baseline conductivity values, especially for the skull which is highly variable.

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1 Introduction

Electrical impedance tomography (EIT) is a non-invasive method of determining the local electrical properties of biological tissues. Impedance measures correlate with a variety of physiological and pathological conditions, and can be used to detect and approximately localize regions of interest. Strictly speaking, EIT is not an imaging modality, at least not in the sense of magnetic resonance imaging (MRI) and x-ray computed tomography (CT), since true tissue boundaries can not be precisely delineated. Speaking mathematically, the equations relating volume impedances to surface potential measurements are nonlinear in the impedance parameters, and can not be inverted to yield explicit solutions. Rather, inverse methods must be applied, and the ill-posed nature of the problem requires additional constraints and methods of iterative refinement.

A major outstanding problem in clinical medicine is the detection and monitoring of brain stroke. The two types of stroke requiring advances are ischemic and hemorrhagic. In ischemic stroke, brain cells are deprived of oxygen, typically due to arterial blockage, resulting in local, gradual cell death via metabolic collapse. This occurs in approximately 700,000 Americans annually, and leaves many dead or paralyzed. Hemorrhagic stroke occurs when blood leaks into the brain compartment, and either puts pressure on the brain, or kills brain cells directly by a toxic effect. Brain hemorrhage can be caused by head impact, in which case the clinician often (but not always) has obvious clues to expect and look for existing hemorrhage. Still, hospitals today do not have a method of continuously monitoring for hemorrhage onset, and it is not uncommon for patients to be released from the hospital only to return home and die without sufficient warning. With cruel irony, brain hemorrhage may also be caused by thrombolytic drugs used to treat cortical ischemia, which carry a 6% risk of causing brain hemorrhage.

The motivation for this paper is that both ischemic and hemorrhagic stroke are accompanied by impedance changes, so the development of EIT for stroke detection and monitoring could significantly improve patient outcomes. A advantage of EIT which provides further motivation is that, unlike x-ray CT and MRI, EIT can be implemented for the head using electroencephalographic (EEG) electrode arrays and amplifiers, which are highly portable and cost effective. Thus EIT can be made available in ambulances and field hospitals, and intensive care units (ICU) where they might be applied to track ischemic progression and recovery, as well as the possible development of brain hemorrhage.

While EIT has been applied extensively to the thorax, where current passes essentially unhindered through the rib cage, it has only barely begun to be studied in the head. The predominant thinking is that, when current is injected into the scalp, the low skull conductivity will cause most current to be shunted through the scalp, preventing surface potential measurements from being sensitive to brain changes. We refute this argument here, primarily because most authors have underestimated the skull conductivity to be 1/80 that of the scalp. A summary of the
data collected by Law (1993) suggest that this ratio is closer to 1/24, but this number may be suspect because the data were collected from a dry skull which was resaturated, and it could not be assured that the conductivity of the skull was reconstituted accurately (Nunez, personal communication). A more recent study suggests a factor closer to 1/30, and this number seems quite reliable as it was determined from fresh human cadavers [17]. Regarding the inverse problem, Fervere et al. [3] have demonstrated using numerical simulations that with appropriate inverse methods it should be possible to measure the average regional conductivity of the brain, CSF, skull and scalp. If the outer tissues were then considered fixed, then changes in the brain might easily be detected by EIT.

It is interesting that sometime quantities which are theoretically well defined become less tangible numerically. This leads to the use of truncated singular value decomposition and other

2 Theory

2.1 Forward problem

The theoretical foundation of EIT is Poisson’s equation for the electric potential \( \Phi \) subject to Neumann boundary conditions:

\[
\vec{\nabla} \cdot (\sigma \vec{\nabla} \Phi) = 0 \quad \text{in } \Omega \quad (1)
\]

\[
\frac{\partial \Phi}{\partial \nu} = -\frac{J_\perp}{\sigma} \quad \text{on } \partial \Omega \quad (2)
\]

where \( \sigma \) is the local tissue conductivity, \( \Omega \) and \( \partial \Omega \) denote head volume and boundary, respectively, \( \nu \) is the outward pointing normal vector on \( \partial \Omega \), and \( J_\perp \) is the normal current density evaluated at the surface \( \partial \Omega \).

2.2 Reduction to two dimensions

The problem of interest is the 3-D head volume, and numerical studies have shown that a 2-D model is not adequate to compute surface potentials accurately [ ... ]. Even if the volume conductivity has translational uniformity, a 2-D model is still only approximate [12]. Yet it is common to use 2-D models as a starting point, since their solution is simpler, computationally more efficient, and easier to visualize.

The algorithms developed in two dimensions may be put into relation with the corresponding three-dimensional problem as follows. In cylindrical coordinates, by separation of variables the solution to (1) may be written

\[
\Phi(r, \theta, z) = \sum_{k=0}^{\infty} \phi_k(r, \theta) \cos \left( \frac{k\pi}{L} z \right) \quad (3)
\]

for an object of vertical dimension \( z \in [-L, +L] \). Inserting this expression into the first equation in (1) leads to

\[
\vec{\nabla}_z \cdot (\sigma \vec{\nabla}_z \phi_k) - \sigma \left( \frac{k\pi}{L} \right)^2 \phi_k = 0 \quad (4)
\]

It is clear that by simply solving Poisson’s equation in 2-D dimensions, only the \( k = 0 \) contribution to the 3-D series solution is obtained. Yet it has been demonstrated in the present context that 2-D and axially symmetric 3-D problems are similarly ill-posed [12]. Most 3-D problems of
interest have no exact symmetry, however, and a more focal source distribution together with more numerous independent measurements require separate study.

2.3 Reciprocity formulation

Our starting point for EIT is the Geselowitz-Lehr theorem [6, 13] relating changes in local conductivity to changes in surface potential. Figure 1 shows the situation.

![Electrode configurations](image)

Figure 1: Electrode configurations relevant to four-electrode reciprocity theorem [9].

At time \( t_1 \), current \( I_{inj} \) is passed through a pair of electrodes AB, and the potential difference \( \Phi_1 \) is measured across a pair CD, assumed to be distinct. At time \( t_2 \), current is passed through CD, and the potential difference \( \Phi_2 \) is measured across AB. The mutual impedance is defined \( Z_i \equiv \Phi_i/I_{inj} \). If the conductivity changes from \( \sigma_1 \) to \( \sigma_2 \), then it is straightforward to show that the mutual impedance changes according to

\[
Z_2 - Z_1 = - \int \Delta \sigma \, \vec{L}_1(\sigma_1) \cdot \vec{L}_2(\sigma_1 + \Delta \sigma) \, d\Omega
\]

\[
\simeq - \int \Delta \sigma \, \vec{L}_1(\sigma_1) \cdot \vec{L}_2(\sigma_1) \, d\Omega
\]

where \( \Delta Z \equiv Z_2 - Z_1 \), and \( \Delta \sigma \equiv \sigma_2 - \sigma_1 \) is assumed small. The lead field vector \( \vec{L}_i \equiv - \nabla \Phi_i/I_{inj} \) is proportional to the local current density \( \vec{J} \) obtained when unit current is injected into pair \( i \). Note that this formulation does not rely upon the potential across the injection electrodes, which is difficult to determine because the scalp-electrode impedance is not usually known [8].

In electrical impedance tomography (EIT) the above result is applied iteratively to solve the full nonlinear problem. We define a measurement as a set of four distinct electrodes: \( i \equiv \{ A, B; C, D \} \). The current \( I_{AB} \) is injected through the former, and the potential difference \( \Delta \Phi_{CD} \equiv \Phi_C - \Phi_D \) is measured across the latter.

Equation (5) is nonlinear in \( \Delta \sigma \) by virtue of its dependence inside \( L_i(\sigma + \Delta \sigma) \). This may be handled by first linearizing

\[
\Delta Z_i \simeq - \int \Delta \sigma \, |L_i(\sigma)|^2 \, d\Omega
\]

The FE method discretizes the volume into \( j = 1, \ldots, N \) elements. Assuming \( \Delta \sigma \) is constant over each element, the integral may be written as a sum:

\[
\Delta Z_i \simeq \sum_{j=1}^{N} S_{ij} \, \Delta \sigma_j
\]

where the sensitivity matrix \( S_{ij} \) is defined

\[
S_{ij} \equiv - \int \frac{|L_i(\sigma_j)|^2 \, d\Omega}{\Omega_j}
\]

This linear formulation is only approximate, valid for small \( \Delta \sigma \). Moreover, the sensitivity matrix (8) tends to be singular, so that the corresponding inverse problem of solving (7) for \( \Delta \sigma_j \) is ill-posed and requires regularization.
2.4 Independent measurements

Consideration of the number of linearly independent measurements $M$ which contribute to $S_{ij}$ is central to the main result of this paper. One way to count them is the following. Given a total number $N_e$ electrodes, there are at most $N_p = N_e(N_e - 1)/2$ unique electrode pairs available for current injection. Due to the usual difficulties introduced by the unknown skin-electrode impedance [8], potentials at injection electrodes are not included in the analysis, leaving only $(N_e - 2)$ electrodes to measure voltage. Since one electrode must be used as a reference, there are at most $(N_e - 3)$ independent voltage measurements. Taking all injection and measurement pairs gives $M_i = N_p(N_e - 3)$ independent impedance measurements theoretically. For $N_e = 16$ surface electrodes, this implies $N_p = 120$ injection pairs, and $M_i = 1560$ total possible measurements.

In most papers on EIT [ ... ], this number is reduced significantly by the four-electrode reciprocity theorem [9], which states that the measured potential in EIT, for any measurement set $i$, is preserved under an interchange of injection set $AB$ and measurement set $CD$ electrode pairs. This relation is apparent in (7) since $\Delta Z_i \rightarrow 0$ as $\Delta \sigma_j \rightarrow 0$. If this reciprocity argument is applied in the OPP method (defined below), then effectively there are not $N_p$ but only $N_e/2$ independent injection pairs, resulting in $M_i = (N_e/2)(N_e - 3)$ independent measurements. For $N_e = 16$, this implies $N_e/2 = 8$ injection pairs, and $M_i = 104$ total possible measurements. Clearly this speeds data acquisition, which can be the limiting factor in practical applications.

Despite the fact that most authors assume reciprocity is satisfied, i.e., use 104 independent measurements for 16 electrodes, a variety of EIT measurement schemes have appeared in the literature [14], which are theoretically equivalent by reciprocity, yet seem to have found their niches in application. In the “adjacent” or “neighboring” pattern (ADJ), the injection and measurement pairs are each composed of adjacent electrodes, and the two pairs are moved independently around the disc [16]. In the “opposite” or “diametric” pattern (OPP), the injection pairs are composed of diametrically opposed electrodes, but the measurement pairs are composed of adjacent electrodes as before [10]. We consider ADJ and OPP, along with the “all” pattern with $M_i = N_p(N_e - 3)$ total measurements.

By using reciprocity and re-referencing sequentially, it is possible to show the theoretical equivalence between the ADJ and OPP measurement patterns. Yet this equivalence may be violated in the presence measurement noise. Indeed, several authors have noted that the different measurement patterns do have different effective sensitivities, and that this choice does affect image quality [1, 2, 14, 21, 22]. There have also been several attempts in the literature to derive optimal measurement schemes [ ... ]. The former presumes a cosine-like distributed current injection pattern, to cast the EIT problem to be more like X-ray computed tomography (CT), by seeking straight equipotential lines through the volume [ ... ]. This is obtainable analytically for a homogeneous conducting sphere, but seems unlikely to be obtainable in this problem, due to
the low skull conductivity.

2.5 Matrix conditioning

The sensitivity matrix reconstruction tends to favor near surface structures, and thus the deeper anomalies are blurred or not detected. One way to lessen this effect is to introduce a weighting scheme which normalizes each element for the average sensitivity to the $M$ measurements [7, 18]. Equation (7) may always be rewritten as

$$\Delta Z^{(n)} = (S\ W)\ (W^{-1}\Delta\sigma^{(n)})$$  \hspace{1cm} (9)$$

provided the matrix $W$ is non-singular, i.e., has a unique and well-behaved inverse. This weighting procedure is useful if the matrix $SW$ is better conditioned than $S$.

The simplest way to ensure that $W$ is non-singular is to make it diagonal and positive definite:

$$W = \text{diag}(1/w_j)$$ \hspace{1cm} (10)$$

A good choice for the weights is

$$w_j = \sqrt{\sum_{i=1}^{M} S_{ij}^2}$$ \hspace{1cm} (11)$$

This matrix $W$ normalizes $S$ so that, on average over the $M$ measurements, the contributions from different depths are equal. This is analogous to the weighting scheme in the familiar “weighted minimum norm” algorithm in EEG [17]. To our knowledge this is the first time this weighting scheme has been applied to EIT. A major point of this paper is to show that it acts to balance the differences between the various measurement patterns, and improves image quality overall.

2.6 Iterative nonlinear solution

Given an approximate solution to the linear inverse problem (7), e.g., via the Moore-Penrose pseudoinverse $S^+$ described below, the desired solution to the nonlinear problem (5) may be obtained formally using an iterative approach [16]

$$\sigma_j^{(n+1)} = \sigma_j^{(n)} + \eta \sum_{i=1}^{M} (S^+)_ji \left[ \hat{Z}_i - Z_i^{(n)} \right]$$ \hspace{1cm} (12)$$

where $\Delta Z^{(n)}$ is taken to be the difference between measured $\hat{Z}_i$ and computed $Z_i^{(n)}$ impedances, and $n$ is the iteration step number. The parameter $\eta \in (0, 1]$ is chosen to improve convergence. We found that $\eta = 0.33$ worked well for this problem, however, the choice of such parameters is typically subjective and problem dependent. For a convergence criteria, we choose to terminate when

$$\sum_{i=1}^{M} \left( \Delta Z_i^{(n)} \right)^2 < M\alpha\mu^2$$ \hspace{1cm} (13)$$

where $\mu$ is the noise level and $\alpha$ is a regularization parameter described in [18]. This convergence criteria is derived in [18] and corresponds to stopping the iterative procedure when the difference between measured and computed impedances is less than the noise. Typically about 10 to 15 iterations were needed to reach this level.
3 Numerical results

3.1 Finite element method

We solve the above equation using the Finite Element Method (FEM). FEM is widely used to solve partial differential equations [20], and is described here only briefly. The region $\Omega$ is discretized into small volume elements, each characterized by a single, scalar conductivity $\sigma$. This ultimately transforms the governing system (1) into a discrete linear system:

$$\mathbf{c} = \mathbf{Y}\mathbf{v}$$  \hspace{1cm} (14)

where $\mathbf{c}$ is the vector of injected currents at each node, $\mathbf{v}$ is the vector of potentials at each node, and $\mathbf{Y}$ is called the global stiffness matrix.

The global stiffness matrix is computed as a sum of local contributions, as follows. For a two dimensional mesh with a single element, the global stiffness matrix is $3 \times 3$, and the $ij^{th}$ entry is [20]:

$$Y_{ij} = \frac{\sigma}{4A}(b_i b_j + c_i c_j)$$  \hspace{1cm} (15)

where $A =$ area of triangle, $b_i = y_3 - y_2$ and $c_i = x_2 - x_3$, where $x_i$ and $y_i$ refer to the local coordinate of the $i^{th}$ node. The rest of the $b_i$ and $c_i$ can be found by permuting the indices. Equation (15) is only for a one element mesh. In order to incorporate more elements, the local indices $i_L$ are transformed into global indices $i_G$ and then the global matrix $\mathbf{Y}_G$ is computed by adding the local matrices $\mathbf{Y}_L$ accordingly. The resulting matrix is sparse, but not band-diagonal since as formulated the numbering of elements is not so advantageous.

As formulated thus far, $\det \mathbf{Y}_G = 0$ meaning the solution to (14) is not unique. This is expected since Laplace’s equation with Neumann boundary conditions is unique only up to an additive constant. This is remedied by selecting a reference node $i_r$, and setting the $i_r$ row and column to zero, and the diagonal term to one (this forces the corresponding value in $\mathbf{v}$ to zero). Now $\mathbf{Y}_G$ is positive definite and the system may be solved by Cholesky decomposition (CD) or the conjugate gradient (CG) algorithm.

For a two-dimensional FE mesh, assuming that the potential $\Phi$ varies linearly over each element, the sensitivity matrix is [16]:

$$S_{ij} = -\Phi_{AB}^{T} \frac{bb^{T} + cc^{T}}{4A I^2_{inj}} \Phi_{CD}$$  \hspace{1cm} (16)

where $\mathbf{b}$ and $\mathbf{c}$ are the column vectors formed from the global stiffness matrix elements $b_i$ and $c_i$, and $\Phi_{AB}$ and $\Phi_{CD}$ are the column vectors of $\Phi_{AB,i}$ and $\Phi_{CD,i}$ respectively.

3.2 Head and stroke models

To determine simply whether different measurement patterns have different sensitivities, we computed the changes in surface potential for five focal anomalies, meant to mimic the conductivity changes associated with ischemic and hemorrhagic stroke. Figure 1 shows the FE mesh used for these following computations, including 1141 nodes and 2166 triangles. It is similar to the one used by Bayford et al. [1], but with more triangles in the skull layer to better accommodate the
steeper potential gradients. Each stroke region is comprised of 32 elements, and have area equal to 5.47 cm². This mesh allows us to model four layers with baseline parameters given by: \( \sigma_b = 0.15 \) S/m (brain), \( \sigma_c = 1.79 \) S/m (CSF), \( \sigma_s = 0.015 \) S/m (skull), and \( \sigma_t = 0.44 \) S/m (scalp). The brain, CSF and scalp values are motivated as in [3]. The skull value is derived from Oostendorp et al. [17]. Note that the experiments therein suggest \( \sigma_b/\sigma_s \approx 10 \), which is greatly favorable compared to the commonly used \( \sigma_b/\sigma_s \approx 80 \). The confusion which has persisted for some time in the EEG literature is explained clearly in [17]. Note these same parameters imply \( \sigma_b/\sigma_s \approx 30 \), which may be more relevant to EIT, as it determines the amount of shunting through the scalp.

Figure 2: Finite element mesh consisting of 1141 nodes and 2166 elements. Outer radii are equal to 8.0 cm (brain), 8.2 cm (CSF), 8.7 cm (skull), and 9.2 cm (scalp).

We assume that hemorrhage may be characterized by a region whose conductivity is equal to that of blood: \( \sigma = 0.61 \) S/m [4], and that ischemic tissue may characterized by a region whose conductivity is 50% that of brain: \( \sigma = 0.075 \) S/m [10]. The size of the anomalies were chosen uniformly, but a more complete study of detection capability should include the dependence on size.

### 3.3 Detection sensitivity

Tables 2 and 3 show the changes in surface potential \( \delta \Phi \equiv \Delta \Phi_{CD}(\sigma + \Delta \sigma) - \Delta \Phi_{CD}(\sigma) \) due to introducing the impedance anomalies in Table 1, when \( I_{\text{inject}} = 100 \, \mu A \) is applied. We compared ADJ and OPP measurement patterns, as well as the ALL pattern based upon \( M_t \) measurements. To provide a more sensible comparison, in the ALL case we report the average over the largest 104 changes only. (If all 1560 differences are computed, then the average values of \( \delta \Phi \) fall between those for ADJ and OPP.)

| TABLE 2: Detected \( \delta \Phi (\mu V) \) – Hemorrhage |
|-------------|----------|--------|--------|--------|
|            | Adjacent |        | Opposite | All Pairs |
| R           | max      | ave    | max     | ave    | max     | ave     |
| 1           | 2.83     | 0.44   | 10.65   | 2.19   | 14.89   | 8.30    |
| 2           | 1.52     | 0.34   | 7.10    | 1.85   | 9.51    | 5.98    |
| 3           | 0.97     | 0.29   | 4.74    | 1.79   | 5.89    | 4.42    |
| 4           | 0.73     | 0.27   | 3.72    | 1.73   | 4.21    | 3.50    |
| 5           | 0.64     | 0.26   | 3.32    | 1.71   | 3.63    | 2.95    |

Clearly the larger the maximum \( \delta \Phi \) the more the impedance change of interest will stand out from the measurement noise. For all anomalies, and both hemorrhage and ischemia, the maximum and average \( \delta \Phi \) are ordered according to ALL > OPP > ADJ. The relation between OPP
and ADJ may explain the result in [10]. The fact that the ALL pattern is most sensitive yet is the main motivation for the studies below, as suggests it may be possible to select optimal injection patterns from this complete set of ALL measurements, which would be particularly sensitive to certain regions of the volume. This argument is much like that made by Gencer et al. [5], in the context of optimal reference electrode placement in EEG. Our restriction to the brain layer selects OPP over ADJ patterns, while a different restriction to the scalp layer might select ADJ over OPP, for example. The effect of this observation on tomographic reconstruction depends on other factors as well.

**TABLE 3: Detected $\delta \Phi$ (µV) – Ischemia**

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<td>1.36</td>
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<td>5.05</td>
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<td>3</td>
<td>0.54</td>
<td>0.15</td>
<td>2.48</td>
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<tr>
<td>4</td>
<td>0.40</td>
<td>0.14</td>
<td>1.92</td>
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<tr>
<td>5</td>
<td>0.34</td>
<td>0.14</td>
<td>1.72</td>
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The detectability of the change $\delta \Phi$ must be considered in relation to the noise levels. Injecting $I_{AB} = 100 \ \mu A$ into our FE model results in $\Delta \Phi_{CD}$ ranging over $\pm 2500 \ \mu V$. The acquired signal is assumed to be a sum of the impressed potential $\Delta \Phi_{CD}$ and two additive noise contributions: 1) amplifier noise results in the addition of a zero-mean, 0.6 $\mu V$ RMS Gaussian random variable to each time point, and 2) resting EEG contributes a broadbanded signal with amplitude as high as 10 $\mu V$. By injecting at 100 Hz for five seconds per pair with 250 Hz sampling, then using the FFT to determine amplitude and phase of the impressed potentials, the retrieval error is approximately Gaussian and less than 0.01 $\mu V$ RMS. Higher frequencies and better signal analytic techniques could further reduce this error and speed acquisition time.

### 3.4 Singular value decomposition

Table 4 lists the maximum singular value $\lambda_{\text{max}}$, condition number Cond $\equiv \log_{10}(\lambda_{\text{max}}/\lambda_{\text{min}})$, and Rank of $\mathbf{S}$, defined as the number of singular values greater than some tolerance $\lambda_{\text{tol}}$, for each pattern, assuming uniform brain conductivity $\sigma = 0.15 \ \text{S/m}$ Ref. [19] suggests using $N$ times the expected noise levels, which would imply $\lambda_{\text{tol}} \approx 0.001 \ \Omega^2 m$, but we obtained better results with $\lambda_{\text{tol}} = 0.01 \ \Omega^2 m$. (Since the singular vectors are normalized to unity, the singular values carry the units of the matrix.)

**TABLE 4: The matrix $\mathbf{S}$**

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<td>Cond</td>
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<td>6.00</td>
<td>6.49</td>
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<tr>
<td>$\lambda_{\text{max}}$</td>
<td>0.13</td>
<td>1.00</td>
<td>3.08</td>
</tr>
<tr>
<td>Rank</td>
<td>24</td>
<td>34</td>
<td>48</td>
</tr>
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The values of $\lambda_{\text{max}}$ and Rank are ordered according to ADJ < OPP < ALL, suggesting very different behavior in the presence of noise. The values of Cond are ordered similarly, which does not work in favor of a well behaved solution, but the differences are well within machine precision, which is the relevant comparison for this parameter. There is not a simple one-to-one relationship between a singular value $\lambda$ and the potential change at across a particular electrode pair, since the singular vectors mix components, but
the contribution of each singular value \( \lambda \) to \( \Delta Z_i \) scales linearly with \( \lambda \), suggesting that larger singular values will provide a better signal-to-noise ratio. This may also partly underly the claim in [1] that OPP is preferable to ADJ.

Table 5 shows the maximum singular value \( \lambda_{\text{max}} \), the condition number Cond, and the Rank of the matrix \( \text{SW} \). Comparison with Table 4 shows that the main effect is to increase the sensitivity as measured by \( \lambda_{\text{max}} \), and increase the effective Rank, without changing the condition number Cond significantly.

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It seems natural to expect that the tomographic reconstructions obtained from \( \text{SW} \) will be better than those from \( S \), and this is demonstrated numerically in Section 4.

### 3.5 Regularization via SVD

The system in (7) is ill-posed, since \( S_{ij} \) is singular [22]. In the present problem, this arises primarily because there are many more unknowns \( N \) than measurements \( M \). For the mesh in Figure 1, this is improved somewhat by considering the outer tissues (scalp, skull, CSF) fixed. Even if \( M \) were equal to \( N \), this would still not make the problem well-posed, due to the blurring effects of the skull and volume conductor generally, and the presence of measurement noise.

The linear system (7), expressed in matrix notation as

\[
\Delta Z = S \Delta \sigma
\]

may be solved approximately using the method of singular value decomposition (SVD). The non-square matrix \( S \) may be decomposed as \( S = U \Sigma V^T \), where \( \Sigma \) is a diagonal matrix constructed from the ordered set of singular values \( \lambda \). This allows an approximate solution of the form

\[
\Delta \sigma \simeq S^+ \Delta Z
\]

where \( S^+ \) represents the Moore-Penrose pseudoinverse, given by \( S^+ = V \Sigma^{-1} U^T \) [11]. The key to its utility is that \( \Sigma^{-1} \) denotes the inverse of \( \Sigma \), suitably truncated prior to inversion in order to exclude singular values \( \lambda \) which are less than some tolerance.

### 3.6 Tomographic reconstruction

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4 Discussion

Given this noise estimate, which is presumably conservative, it appears that all five anomaly regions considered here should be detectable, admittedly for hemorrhage more than ischemia. This result is due in part to the best current estimates of skull conductivity [17]. Errors in estimating the static conductivities will lead to systematic errors not considered here, but in practice the acute, stroke-related changes in Tables 2 and 3 will appear as temporal trends, and should therefore be separable from static modeling errors.

The truncated SVD approach provides an approximate solution to (7) which has minimum error in the usual squares sense, as well as minimum norm. Another way to solve the least squares problem is the Marquardt method [19, 18, 22]. The Marquardt method replaces (7) by:

$$(S^T S + \mu I) \Delta \sigma = S^T \Delta Z$$  \hspace{1cm} (18)$$

where $I$ is the identity matrix and $\mu$ is a regularization parameter. **What is the relation to SVD exactly?** The Marquardt is widely recognized as superior to the SVD method, since it does not require performing the singular value decomposition, and is therefore computationally more efficient. **Other reasons?**

Ideally the regularization parameter $\mu$ is chosen in some objective way. One technique is described by Portniaguin [18]. Another is provided by Russell et al., in the context of EEG and with a basis in Bayesian theory.

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**References**


