Boundary element solution for head
tissue conductivity estimation

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This report details the implementation of the boundary element method (BEM) for computing scalp potentials due to scalp current injection. The analytic derivation follows most closely that of Claydon et al. (1988). The discrete formulation follows most closely that of Barr et al. (1977).

1 Analytic solution

To compute the scalp potentials generated by scalp current injection, we assumed that the head is comprised of four tissue layers: brain, CSF, skull and scalp, and that each layer is described by a single homogeneous and isotropic conductivity $\sigma$. Assuming no sources inside the head volume, the potential $\Phi_a$ in each tissue layer $a$ obeys Laplace’s equation

$$\nabla^2 \Phi_a = 0$$

(1)

Green’s theorem implies that for a volume $V_a$ bounded by a closed surface $S_a$, the potential $\Phi_a$ at any point $\vec{r}_0$ inside $V_a$ obeys
\[ 4\pi \sigma_a \Phi_a(\vec{r}_o) = \sigma_a \int_{S_a} \Phi_a(\vec{r}) \, d\Omega_{\vec{r},\vec{r}_o} + \sigma_a \int_{S_a} \frac{1}{|\vec{r}_o - \vec{r}|} \left( \vec{\nabla} \Phi_a(\vec{r}) \cdot \hat{n}_a \right) dS \]  \hspace{1cm} (2)

where \( \vec{r} \) is an integration variable that runs over the surface \( S_a \), and

\[ d\Omega_{\vec{r},\vec{r}_o} \equiv \frac{(\vec{r} - \vec{r}_o) \cdot \hat{n}_a}{|\vec{r} - \vec{r}_o|} \, dS \]  \hspace{1cm} (3)

denotes the solid angle subtended by the surface element \( dS \) at \( \vec{r} \) as viewed from the point of observation \( \vec{r}_o \). The unit vector \( \hat{n}_a \) is normal to the surface element \( dS \) on \( S_a \) and points out of \( V_a \). If \( \vec{r}_o \) is outside \( V_a \) then the left side of (2) is zero.

Equation (2) holds for each tissue layer \( a \). Summing the corresponding equations for each of the four tissue layers leads to a single equation which is valid for \( \vec{r}_o \) located anywhere in the head volume (Geselowitz, 1967). Since the left hand side of (2) is nonzero only if the point of observation \( \vec{r}_o \) is in \( V_a \), after summation we have

\[ 4\pi \sigma_0 \Phi_0(\vec{r}_o) = \sum_{a=1}^{4} \left[ \sigma_a \int_{S_a} \Phi_a(\vec{r}) \, d\Omega_{\vec{r},\vec{r}_o} + \sigma_a \int_{S_a} \frac{1}{|\vec{r}_o - \vec{r}|} \left( \vec{\nabla} \Phi_a(\vec{r}) \cdot \hat{n}_a \right) dS \right] \]  \hspace{1cm} (4)

where \( \sigma_a \) and \( \Phi_a \) are to be thought of as piecewise continuous functions taking on values appropriate to the location of \( \vec{r}_o \).

For all but the innermost layer, the surface \( S_a \) bounding tissue layer \( a \) is multiply connected. This means that on the boundary between layers 1 and 2, for example, \( \hat{n}_1 \) points outward while \( \hat{n}_2 \) points inward. It is therefore useful to define \( \Gamma_a \) as the outer boundary of layer \( a \), and use \( \hat{\gamma}_a \) to denote its outward unit normal. Then for each layer \( \hat{\gamma}_a \) points in the outward direction. At each boundary \( \Gamma_a \), we require continuity of the potential

\[ \Phi_a \bigg|_{\Gamma_a} = \Phi_{a+1} \bigg|_{\Gamma_a} \]  \hspace{1cm} (5)

and continuity of the normal current density

\[ \sigma_a \vec{\nabla} \Phi_a \cdot \hat{\gamma}_a \bigg|_{\Gamma_a} = \sigma_{a+1} \vec{\nabla} \Phi_{a+1} \cdot \hat{\gamma}_a \bigg|_{\Gamma_a} \]  \hspace{1cm} (6)

where \( a = 1, 2, 3 \) and \( a+1 \) refers to the layer just outside layer \( a \). The remaining boundary condition specifies the potential or normal current density on the outer surface \( \Gamma_4 \), and will be dealt with later. Conditions (5) and (6) can be used to simplify (4) to yield
\[ 4\pi\sigma_o \Phi_o(\vec{r}_o) = \sum_{a=1}^{3} \left[ (\sigma_a - \sigma_{a+1}) \int_{\Gamma_a} \Phi_a(\vec{r}) \ d\Omega_{\vec{r},\vec{r}} \right] \]

\[ + \sigma_4 \int_{\Gamma_4} \Phi_4(\vec{r}) \ d\Omega_{\vec{r},\vec{r}} - \int_{\Gamma_4} \frac{1}{|\vec{r}_o - \vec{r}|} J_\perp(\vec{r}) \ d\Gamma \]  

(7)

where the current density is \( \vec{J}(\vec{r}) \equiv -\sigma_4 \nabla \Phi(\vec{r}) \) with normal component \( J_\perp(\vec{r}) \equiv \vec{J}(\vec{r}) \cdot \hat{\kappa}_4 \).

Technically speaking, (7) is only an integral equation for \( \Phi \) since it involves integrals of both \( \Phi \) and \( \nabla \Phi \) over the surface \( \Gamma_4 \). This amounts to an over-specification of the electrostatic boundary value problem (Jackson, 1975). In the “Dirichlet” problem, the potential \( \Phi \) is specified on the outer surface. In the “Neumann” problem, its normal derivative, or equivalently the normal current density \( J_\perp \), is specified on the outer surface. For our problem we chose the latter, as described below.

\[ 2 \]  

Discrete equations for potentials on vertices

Equation (7) can be solved numerically by first representing each of the boundary surfaces \( \Gamma_b \) in terms of a discrete set of triangles. In practice this is accomplished by continuously deforming a polyhedron, comprised of \( M_b \) triangles and \( N_b = (M_b/2) + 2 \) vertices, to approximate each of the head tissue boundaries. This leads to

\[ 4\pi\sigma_o \Phi_o(\vec{r}_o) = \sum_{b=1}^{3} \left[ (\sigma_b - \sigma_{b+1}) \sum_{\beta=1}^{M_b} \int_{\Delta_\beta} \Phi_b(\vec{r}) \ d\Omega_{\vec{r},\vec{r}_\beta} \right] \]

\[ + \sigma_4 \sum_{\beta=1}^{M_4} \int_{\Delta_\beta} \Phi_4(\vec{r}) \ d\Omega_{\vec{r},\vec{r}_\beta} - \sum_{\beta=1}^{M_4} \int_{\Delta_\beta} \frac{1}{|\vec{r}_o - \vec{r}_\beta|} J_\perp(\vec{r}) \ d\Gamma_4 \]  

(8)

Note that on the right side the subscript \( \beta \) has been added to \( \vec{r} \) to make the triangle of integration explicit.

Equation (8) is still an exact result. The fundamental approximation in the BEM is to simplify the surface integrals by factoring out terms that depend only on the geometry. In its simplest formulation, this is accomplished by assuming that the potential and current
density are constant and equal to their average values over the surface of each triangle. This yields

\[
4\pi\sigma_0 \Phi_o(\bar{r}_o) \simeq \sum_{b=1}^{3} \left( (\sigma_b - \sigma_{b+1}) \sum_{\beta=1}^{M_b} \langle \Phi_b \rangle_\beta \int_{\Delta_\beta} d\Omega_{\bar{r}_o \bar{r}_\beta} \right) \\
+ \sigma_4 \sum_{\beta=1}^{M_4} \langle \Phi_4 \rangle_\beta \int_{\Delta_\beta} d\Omega_{\bar{r}_o \bar{r}_\beta} - \sum_{\beta=1}^{M_4} \langle J_4 \rangle_\beta \int_{\Delta_\beta} \frac{1}{|\bar{r}_o - \bar{r}_\beta|} d\Gamma_4
\]

(9)

where \( \langle \cdot \rangle_\beta \) denotes an average over triangle \( \beta \).

Following Barr et al. (1977) we formulated the boundary element problem in terms of the potentials on the vertices (as opposed to the average potentials on the triangles). Matrix equations were generated by letting the point of observation move over the inside of each of the surfaces \( \Gamma_b \). Writing one equation for each vertex on each surface leads to \( N_{\text{tot}} = \sum_{b=1}^{4} N_b \) coupled equations:

\[
4\pi\sigma_a \Phi_a(\bar{r}_a) \simeq \sum_{b=1}^{3} \left( (\sigma_b - \sigma_{b+1}) \sum_{\beta=1}^{M_b} \langle \Phi_b \rangle_\beta \int_{\Delta_\beta} d\Omega_{\bar{r}_a \bar{r}_\beta} \right) \\
+ \sigma_4 \sum_{\beta=1}^{M_4} \langle \Phi_4 \rangle_\beta \int_{\Delta_\beta} d\Omega_{\bar{r}_a \bar{r}_\beta} - \sum_{\beta=1}^{M_4} \langle J_4 \rangle_\beta \int_{\Delta_\beta} \frac{1}{|\bar{r}_a - \bar{r}_\beta|} d\Gamma_4
\]

(10)

where \( \Phi_a^i \equiv \Phi_a(\bar{r}_i) \). In principle this approximation becomes more valid as the number of triangles is increased. Terms involving integrals of the potentials and normal current density are treated separately below.

### 2.1 Potential terms

To evaluate the integrals involving the potentials, we first separated contributions from “near” and “distant” triangles

\[
\sum_{\beta=1}^{M_b} \langle \Phi_b \rangle_\beta \int_{\Delta_\beta} d\Omega_{\bar{r}_a \bar{r}_\beta} \rightarrow \delta_{ab} \Phi_a^i \hat{\Omega}_a^i + \sum_{\beta=1}^{M_b} \langle \Phi_b \rangle_\beta \left( 1 - D^{i\beta}_{ab} \right) \Omega_a^{i\beta}
\]

(11)

as in (Barr et al., 1977), by defining
\[
D_{i\beta}^{ab} = \begin{cases} 
1 & \text{if vertex } i \text{ on } \Gamma_a \text{ is a corner of triangle } \beta \text{ on } \Gamma_b, \\
0 & \text{otherwise} 
\end{cases} 
\] (12)

In the second term in (11), \( \hat{\Omega}_{i\alpha}^{ab} \) represents the solid angle of a triangle \( \beta \) as viewed from a distant point \( i \). For this we used the exact expression derived in van Oosterom and Strackee (1983). Their expression is not valid, however, when the point \( i \) is one of the corners of triangle \( \beta \), hence \( D_{i\beta}^{ab} \) was introduced to separate these contributions. The first term in (11) represents the effective solid angle of all triangles \( \beta \) for which \( i \) is a corner. Since the total solid angle over a closed surface must be \( 4\pi \), we used

\[
\hat{\Omega}_{i\alpha}^{a} = 4\pi - \sum_{\alpha=1}^{M_a} \left( 1 - D_{i\alpha}^{a\alpha} \right) \Omega_{i\alpha}^{a} 
\] (13)

as in Barr et al. (1977). This yields the correct result in the limiting case that \( \Phi_{i}^{a} \) is constant on \( \Gamma_{a} \).

The average potential \( \langle \Phi_{i} \rangle_{\beta} \) over triangle \( \beta \) was approximated as the average of the potentials at the corners of that triangle

\[
\langle \Phi_{i} \rangle_{\beta} \approx \frac{1}{3} \sum_{j=1}^{N_{b}} D_{i\beta}^{jb} \Phi_{i}^{j} 
\] (14)

Inserting (14) into (11) leads to

\[
\sum_{\beta=1}^{M_{b}} \langle \Phi_{i} \rangle_{\beta} \int_{\Delta_{\beta}} d\Omega_{i} \rightarrow \delta_{ab} \Phi_{i}^{a} \hat{\Omega}_{i}^{a} + \sum_{j=1}^{N_{b}} \left[ \frac{1}{3} \sum_{\beta=1}^{M_{b}} \left( 1 - D_{i\alpha}^{\beta} \right) D_{i\beta}^{jb} \Omega_{i\alpha}^{b} \right] \Phi_{i}^{j} 
\] (15)

A better approximation than (14) would be to interpolate the potential over the triangle (Oostendorp and van Oosterom, 1991), but this is deferred for a later study.

2.2 Current density term

To evaluate the integrals involving the current density, we first defined

\[
\sum_{\beta=1}^{M_{4}} \langle J_{\perp} \rangle_{\beta} \int_{\Delta_{\beta}} \frac{1}{|\vec{r}_{i} - \vec{r}_{\beta}|} d\Gamma_{4} = \sum_{\beta=1}^{M_{4}} P_{i\beta}^{j} \langle J_{\perp} \rangle_{\beta} 
\] (16)

where the geometric factor
\[ P_{\alpha 4}^{i,j} \equiv \int_{\Delta_{i,j}} \frac{1}{|\vec{r}_i - \vec{r}_{\beta}|} \, d\Gamma_4 \]  

(17)

To simplify the problem further, the integrand of \( P_{\alpha 4}^{i,j} \) was assumed to be constant over the surface of triangle \( \beta \)

\[ P_{\alpha 4}^{i,j} \approx \left\langle \frac{1}{|\vec{r}_i - \vec{r}_{\beta}|} \right\rangle_{\beta} \int_{\Delta_{i,j}} d\Gamma_4 \]  

(18)

and equal to (the reciprocal of) the average distance from vertex \( i \) on \( \Gamma_{a} \) to corner \( j \) of triangle triangle \( \beta \) on \( \Gamma_{4} \)

\[ \left\langle \frac{1}{|\vec{r}_i - \vec{r}_{\beta}|} \right\rangle_{\beta} \approx \left[ \sum_{j=1}^{N_4} |\vec{r}_i - \vec{r}_{\beta}| \, D_{bb}^{i,j} \right]^{-1} \]  

(19)

The remaining factor in (18) is simply the area of triangle \( \beta \) which can be computed directly. Since (19) is well-behaved even when \( i \) is a corner of triangle \( \beta \), a separation like that used in (11) is unnecessary. A better approximation for (17), however, may be the analytic expression given in Rao, et al. (1979) or in Okon and Harrington (1982). This is deferred for future work.

For simplicity, we assumed Neumann boundary conditions on the outer surface. This is unlike the mixed boundary condition treatments given in Claydon (1988) and Oostendorp and van Oosterom (1991), and becomes most accurate in the limit that the injection electrodes are small compared to the radius of the sphere. We further assumed that each injection electrode \( e \) is situated on a vertex on \( \Gamma_{4} \), and that a net current \( I_{e} \) is injected through that electrode. The average current density over triangle \( \beta \) was approximated by

\[ \langle J_{\perp} \rangle_{\beta} \approx I_{e} \, D_{44}^{i,j} \left[ \sum_{\alpha=1}^{N_4} A_{\alpha} \, D_{44}^{\alpha \beta} \right]^{-1} \]  

(20)

where \( A_{\alpha} \) is the area of triangle \( \alpha \) on \( \Gamma_{4} \). The matrix element \( D_{44}^{i,j} \) ensures that \( \langle J_{\perp} \rangle_{\beta} \) is zero unless vertex \( e \) is a corner of triangle \( \beta \), and the denominator normalizes the total current by the total area of all such triangles. (By convention outward current flow is positive.)
3 Numerical solution

To obtain the numerical solution of (10), the $N_{\text{tot}} = \sum_{b=1}^{4} N_{b}$ equations were concatenated to make a single matrix equation of dimension $N_{\text{tot}}$. Using the variables $n$ and $m$ to index vertices from 1 to $N_{\text{tot}}$, it is possible to write (10) in the form

$$\sum_{m=1}^{N_{\text{tot}}} A_{nm} \Phi_{m} = B_{n}$$

which can be inverted to yield

$$\Phi_{n} = \sum_{m=1}^{N_{\text{tot}}} (A^{-1})_{nm} B_{m}$$

These equations were implemented in the C programming language, and solved numerically using singular value decomposition Press et al. (1992). For a single spherical tissue layer, the potentials obtained agree well with known analytic solutions. Extensions to realistically shaped heads with realistic conductivities are in progress.

References


