Spline Interpolation of the Scalp EEG
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Modern dense-array EEG recording systems make use of 129 electrodes, with 128 amplifier channels recording 128 voltages \( V_i \) relative to a single common reference electrode. In order to render these potentials graphically, one needs an interpolation scheme for estimating scalp potentials between electrode locations. These interpolated potentials can also be used to compute the scalp surface Laplacian, which gives estimates of the local, two-dimensional divergence of the electric current density \( J \) on the surface of the scalp. Taken together, these methods allow visually informative spatial maps of scalp EEG data.

Before describing the different schemes for computing spline interpolations, we must first settle an issue about the reference electrode. EEG amplifiers measure 128 voltages relative to a reference electrode, usually taken to be located at the head vertex. The resulting voltages have the undesired effect that the signal in each electrode includes the intrinsic activity at the vertex. Although not a problem for source localization, to interpret EEG data visually, it is desirable to eliminate this reference electrode effect by computing the average potential over the surface of the scalp and using this as a “quiet reference.” When the electrode potentials are expressed relative to this “average reference” they approximate the “potentials relative to infinity” or “absolute potentials” which are usually emphasized in physics approaches. (For details on computing the average reference correctly, see the EGI Technical Note AverageReference.pdf.) The question arises here whether to interpolate based on the measured potentials or these transformed potentials. Because the interpolated potentials depend linearly on the measured potentials, one can perform the average reference transformation either before or after interpolation. However, since the interpolated potentials can be used to better approximate the average reference, it is more sensible to compute the interpolated potentials directly from the measured data. Thus the voltages below are best thought of as referring to the measured potentials.

There are two spline interpolation schemes used commonly in EEG research. The first are spherical splines (Perrin et al., 1989; Perrin et al., 1990), which assume spherical head geometry. The second are three-dimensional splines, introduced by Perrin et al. (1987) in the context of two-dimensional polar projections of EEG data, then extended to full three-dimensional geometry by Law et al. (1993), and Srinivasan et al. (1996).

Spherical splines

Spherical splines were first introduced to EEG research by Perrin et al. (1989). (Be careful! There are typographical errors in that paper, which were later corrected in Perrin et al. (1990).) To define the spherical splines, let \( V(\mathbf{r}) \) be the potential at an arbitrary point \( \mathbf{r} \) on the surface of a sphere of radius \( r \), and let \( \mathbf{r}_i \) be the location of one of the 128 measurement electrodes. (Note that in this text we use boldface notation to represent vectors, while in the equations we use arrows.) Spherical splines assume that the potential at any point \( \mathbf{r} \) on the surface of the sphere can be represented by
where the function $g_m(x)$ is given by

$$g_m(x) = \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)m} P_n(x)$$

The functions $P_n(x)$ are the Legendre polynomials of order $n$, which form a complete set of basis functions on a spherical surface assuming azimuthal symmetry. (This does not mean that an interpolation based on these functions assumes azimuthal symmetry of the scalp potentials! Because the variable $x$ in $P_n(x)$ represents the angle between electrode position $r_i$ and the interpolation point $r$, this set of functions is capable of describing arbitrary scalp potentials. The procedure works by effectively treating each electrode as a field source located at the north pole, then rotating the coordinate system to account for the actual electrode position.) Figure 1 shows the first seven Legendre polynomials as a function of $x$, where $-1 < x < +1$. The higher orders $n$ include higher polynomial powers of the argument $x$, and therefore oscillate more.

![Figure 1. The first seven Legendre polynomials.](image)

The constant $m$ in $g_m(x)$ may be called the “order” of the interpolation. Notice that higher orders cause the denominator in the sum to grow more rapidly as a function of $n$, hence higher order interpolations emphasize lower order Legendre polynomials. In practice, the sum in $g_m(x)$ must be truncated to some finite number of terms. For $m=1$, the function $g_m(x)$ converges very slowly as a function of the number of terms included in the sum. In contrast, for higher $m$, the sum converges more rapidly. We therefore only recommend using $m \geq 2$. Perrin et al. (1989) stated that in their simulations, for which they chose $m=4$, including only the first seven terms in the sum was adequate to obtain an accuracy of $10^{-6}$. We have verified that seven terms are adequate for any $m \geq 2$. It is interesting to note that for any $m \geq 3$, the function $g_m(x)$ converges to an almost exactly linear function of $x$. In other words, the spherical spline interpolation algorithm for $m \geq 3$ is essentially a linear interpolation algorithm, if the linear interpolation variable $x$ is understood to be the cosine of the angle between each measurement electrode $r_i$ and the interpolation point $r$. 
To adapt the spherical spline expansion to interpolate a particular data set, the constant coefficients $c_0$ and $c_i$, for $i=1,...,128$, are determined by requiring that the function $V(r)$ equal the measured potentials $V_i$ when evaluated at the measurement electrode locations $r_i$, and an additional constraint that the sum of all the coefficients sum to zero. For the exact form of the resulting matrix equation, see Perrin et al. (1989). To invert the matrix, we use singular value decomposition (Press et al., 1992).

![Figure 2. Spherical spline interpolated potentials for 128-electrode data, generated in Phantom Head 2.0, assuming four independent dipoles in the head.](image)

Comparison of the different spherical spline interpolation orders, provided in Figure 2, shows that, although the $m=1$ has some artifacts, the $m=2$ and $m=3$ results are very similar. Further investigation of higher orders $m$ shows nearly identical results, consistent with the realization that fewer and fewer Legendre polynomials contribute to the sum, and that the interpolation becomes effectively linear in the cosine variable $x$ for $m \geq 3$.

**Three-dimensional splines**

An arguably more general class of spline interpolating function was introduced to EEG research by Perrin et al. (1987). They wrote

$$V_m(\bar{r}) = Q_{m-1}(\bar{r}) + \sum_{i=1}^{128} P_i K_{m-1}\left(\frac{1}{2}(\bar{r} - \bar{r}_i)^2\right)$$

where the basis function

$$K_{m-1}\left(\frac{1}{2}(\bar{r} - \bar{r}_i)^2\right) = \left(\frac{1}{w^2}\right)^{m-1} \log\left(\frac{1}{w^2}\right)$$

and $m$ is again called the interpolation order. Here, however, the word “order” has a slightly different meaning than for spherical splines, because the functional forms are different. The constant parameter $w$ is usually taken to be the electrode radius (Perrin et al., 1987), and has the effect of keeping the argument of the log from approaching zero; we return to this point below.

In Perrin et al. (1987), the function $Q_{m-1}$ was defined as a function of two variables only. This is because their original procedure involved first projecting the scalp data onto a circular area on the $xy$-plane, then interpolating the data in this two-dimensional space, and finally rendering this
interpolated data back on the spherical head surface. Hence the vectors \( \mathbf{r} \) and \( \mathbf{r}_i \) appearing in \( K_{m-1} \) represented two-dimensional vectors in the \( xy \)-plane. Similarly, the function \( Q_{m-1} \) was defined as a function of \( x \) and \( y \) only. In later applications (Law et al., 1993; Srinivasan et al., 1996), this function was generalized to three dimensions which allows interpolation on the head surface directly, without projection onto the \( xy \)-plane. For \( m=1 \), \( Q_{m-1} \) has only a single constant term; for \( m=2 \), linear terms are included; for \( m=3 \), quadratic terms are included, etc.

As for the spherical splines, the unknown coefficients \( P_i \) and \( q_{dkg} \) must be fit to the data by requiring that the function \( V(\mathbf{r}) \) equal the measured potentials \( V_i \), when evaluated at the electrode positions \( \mathbf{r}_i \). Unlike for spherical splines, however, here the number of unknown coefficients to be fit increases with increasing interpolation order \( m \). Hence one can expect that the interpolation may be more flexible, but also that the matrix inversion step may become more difficult, and perhaps less numerically stable, with increasing orders. This is even more so because the function \( K_{m-1} \) depends strongly on the order \( m \).

Notice that in the function \( K_{m-1} \) above, we introduced an additional factor of \( w^2 \) in the denominator, of the log, which was not present in any of the previous papers. Despite its absence in these previous papers, this factor is essential for a rigorous interpretation of the function. This is because of the general mathematical point that the argument of a log function must be unitless. One way to see this is to recognize that the log function, like the sine, cosine and exponential functions, has a Taylor series with infinitely many terms, each with increasingly higher powers of the argument:

\[
\log(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \ldots
\]

Hence the log of a quantity which has units has ever-increasing powers of those units, which is clearly unphysical. This becomes particularly apparent when one computes the surface Laplacian from the spline function and tries to keep track of the units. Introduction of the additional factor \( w^2 \) in the denominator fixes this problem, making the calculation of the surface Laplacian in proper units straightforward. It is worth mentioning, however, that any factor with units of length could have been used to normalize the argument. The choice of the electrode diameter is convenient, because it is common to choose a factor which is indicative of some characteristic length scale in the problem, and because this quantity was already being introduced in the numerator by the previous authors. On a spherical head surface one could potentially use the head radius instead, but for realistic head geometry no constant head radius is defined, hence the electrode diameter is the obvious choice. As a simple matter of convenience, we also introduced factors of \( w^2 \) in the polynomial part of the function \( K_{m-1} \), and corresponding factors of \( w \) in the function \( Q_{m-1} \). Since \( K_{m-1} \) is unitless, the coefficients \( P_i \) have units of volts (or \( \mu V \)), regardless of the order \( m \). The coefficients \( q_{dkg} \) also each have units of volts (or \( \mu V \)).
Figure 3 shows the result of interpolating model EEG with three-dimensional splines. For most direct comparison with the spherical splines, the interpolation was performed on a sphere then projected onto the realistic head surface. We see that the results do not depend sensitively on the order. This is good, for the same reason as for the spherical splines of order greater than 2: we like to be able to interpolate data and not have the results not depend sensitively on the choice of order, at least for some range of orders. We also see that that, for \( m=2 \) and \( m=3 \), the results are nearly identical to those shown in Figure 2. This is also good: we like to be able to interpolate data and not have the result depend sensitively on the method.

![Figure 3](image)

**Figure 3.** Three-dimensional spline interpolated potentials for 128-electrode data, generated in Phantom Head 2.0 assuming four independent dipoles in the head.

We do not consider here the effects of interpolating on non-spherical geometry, although we expect that the results would be better for three-dimensional splines than for spherical splines.

**References**


